dynamics of Minkowski and others. For further information on this rather large and important subject, the reader can consult the literature cited at the end of the chapter.

11.10 *Transformation of Electromagnetic Fields*

Since the fields **E** and **B** are the elements of a second-rank tensor $F^{\alpha\beta}$, their values in one inertial frame K' can be expressed in terms of the values in another inertial frame K according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{11.146}$$

In the matrix notation of Section 11.7 this can be written

$$F' = AF\tilde{A} \tag{11.147}$$

where F and F' are 4×4 *matrices* (11.137) and A is the Lorentz transformation matrix of (11.93). For the specific Lorentz transformation (11.95), corresponding to a boost along the x_1 axis with speed $c\beta$ from the unprimed frame to the primed frame, the explicit equations of transformation are

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta B_3) & B_2' &= \gamma(B_2 + \beta E_3) \\ E_3' &= \gamma(E_3 + \beta B_2) & B_3' &= \gamma(B_3 - \beta E_2) \end{aligned} \tag{11.148}$$

Here and below, the subscripts 1, 2, 3 indicate ordinary Cartesian spatial components and are not covariant indices. The inverse of (11.148) is found, as usual, by interchanging primed and unprimed quantities and putting $\beta \rightarrow -\beta$. For a general Lorentz transformation from K to a system K' moving with velocity $\mathbf{v}$ relative to K, the transformation of the fields can be written

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \end{aligned} \tag{11.149}$$

These are the analogs for the fields of (11.19) for the coordinates. Transformation (11.149) shows that **E** and **B** have no independent existence. A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame. Of course certain restrictions apply (see Problem 11.14) so that, for example, a purely electrostatic field in one coordinate system cannot be transformed into a purely magnetostatic field in another. But the fields are completely interrelated, and one should properly speak of the electromagnetic field $F^{\alpha\beta}$, rather than **E** or **B** separately.

If no magnetic field exists in a certain frame K', as for example with one or more point charges at rest in K', the inverse of (11.149) shows that in the frame K the magnetic field **B** and electric field **E** are linked by the simple relation

$$\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E} \tag{11.150}$$

Note that $\mathbf{E}$ is not the electrostatic field in K', but that field transformed from K' to K.

As an important and illuminating example of the transformation of fields, we consider the fields seen by an observer in the system K when a point charge q moves by in a straight-line path with a velocity $\mathbf{v}$. The charge is at rest in the system K', and the transformation of the fields is given by the inverse of (11.148) or (11.149). We suppose that the charge moves in the positive x_1 direction and that its closest distance of approach to the observer is b. Figure 11.8 shows a suitably chosen set of axes. The observer is at the point P. At $t = t' = 0$ the origins of the two coordinate systems coincide and the charge q is at its closest distance to the observer. In the frame K' the observer's point P, where the fields are to be evaluated, has coordinates $x_1' = -vt', x_2' = b, x_3' = 0$, and is a distance $r' = \sqrt{b^2 + (vt')^2}$ away from q. We will need to express r' in terms of the coordinates in K. The only coordinate needing transformation is the time $t' = \gamma[t - (v/c^2)x_1] = \gamma t$, since $x_1 = 0$ for the point P in the frame K. In the rest frame K' of the charge the electric and magnetic fields at the observation point are

$$E_1' = -\frac{qvt'}{r'^3}, \qquad E_2' = \frac{qb}{r'^3}, \qquad E_3' = 0$$

$$B_1' = 0, \qquad B_2' = 0, \qquad B_3' = 0$$

In terms of the coordinates of K the nonvanishing field components are

$$E_1' = -\frac{q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_2' = \frac{qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{11.151}$$

Then, using the inverse of (11.148), we find the transformed fields in the system K:

$$\begin{aligned} E_1 &= E_1' = -\frac{q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \\ E_2 &= \gamma E_2' = \frac{\gamma qb}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \\ B_3 &= \gamma\beta E_2' = \beta E_2 \end{aligned} \tag{11.152}$$

with the other components vanishing.

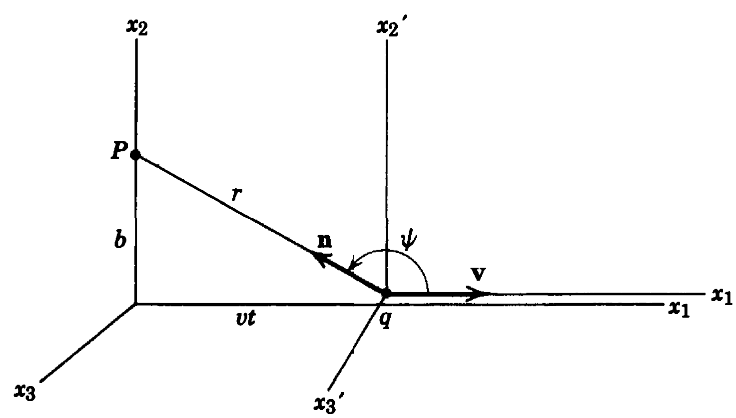

Figure 11.8 Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

Fields (11.152) exhibit interesting behavior when the velocity of the charge approaches that of light. First of all there is observed a magnetic induction in the x_3 direction already displayed in (11.150). This magnetic field becomes almost equal to the transverse electric field E_2 as $\beta \rightarrow 1$. Even at nonrelativistic velocities where $\gamma \simeq 1$, this magnetic induction is equivalent to

$$\mathbf{B} \simeq \frac{q}{c} \frac{\mathbf{v} \times \mathbf{r}}{r^3}$$

which is just the approximate Ampère–Biot–Savart expression for the magnetic field of a moving charge. At high speeds when $\gamma \gg 1$ we see that the peak transverse electric field E_2 $(t = 0)$ becomes equal to γ times its nonrelativistic value. In the same limit, however, the duration of appreciable field strengths at the point P is decreased. A measure of the time interval over which the fields are appreciable is evidently

$$\Delta t \simeq \frac{b}{\gamma v} \tag{11.153}$$

As γ increases, the peak fields increase in proportion, but their duration goes in inverse proportion. The time integral of the fields times v is independent of velocity. Figure 11.9*a* shows this behavior of the transverse electric and magnetic fields and the longitudinal electric field. For $\beta \rightarrow 1$ the observer at P sees nearly equal transverse and mutually perpendicular electric and magnetic fields. These are indistinguishable from the fields of a pulse of plane polarized radiation propagating in the x_1 direction. The extra longitudinal electric field varies rapidly from positive to negative and has zero time integral. If the observer's detecting apparatus has any significant inertia, it will not respond to this longitudinal field. Consequently for practical purposes he will see only the transverse fields. This equivalence of the fields of a relativistic charged particle and those of a pulse of electromagnetic radiation will be exploited in Chapter 15. In Problem 11.18 the fields for $\beta = 1$ are given an explicit realization.

The fields (11.152) and the curves of Fig. 11.9*a* emphasize the time dependence of the fields at a fixed observation point. An alternative description can be given in terms of the spatial variation of the fields relative to the instantaneous *present position* of the charge in the laboratory. From (11.152) we see that $E_1/E_2 = -vt/b$. Reference to Fig. 11.8 shows that the electric field is thus directed along $\mathbf{n}$, a unit radial vector from the charge's present position to the observation point, just as for a static Coulomb field. By expressing the denominator in (11.152) in terms of r, the radial distance from the present position to the observer, and the angle $\psi = \cos^{-1}(\mathbf{n} \cdot \hat{\mathbf{v}})$ shown in Fig. 11.8, we obtain the electric field in terms of the charge's present position:

$$\mathbf{E} = \frac{q\mathbf{r}}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} \tag{11.154}$$

The magnetic induction is given by (11.150). The electric field is radial, but the lines of force are isotropically distributed only for $\beta = 0$. Along the direction of motion ($\psi = 0, \pi$), the field strength is down by a factor of γ^{-2} relative to isotropy, while in the transverse directions ($\psi = \pi/2$) it is larger by a factor of γ. This whiskbroom pattern of lines of force, shown in Fig. 11.9*b*, is the spatial "snapshot" equivalent of the temporal behavior sketched in Fig. 11.9*a*. The compres-

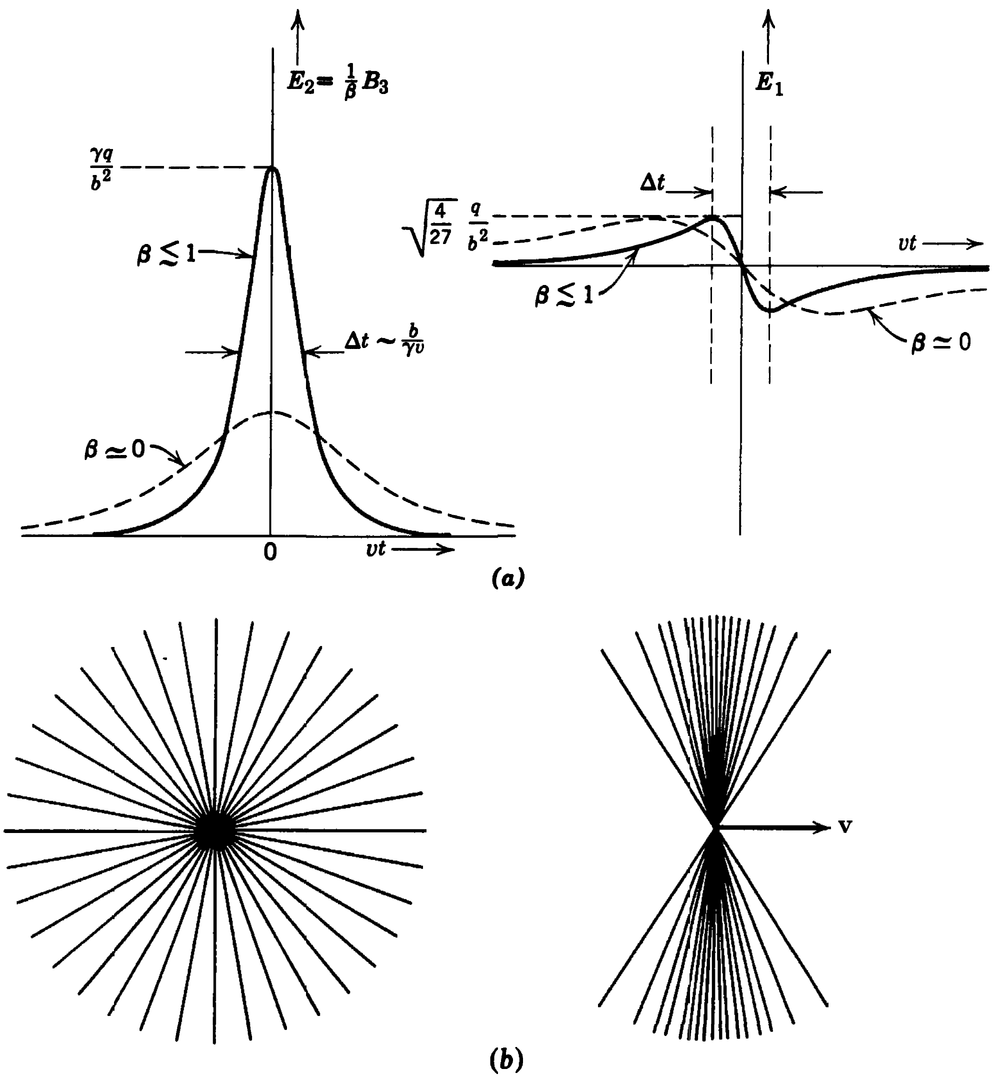

Figure 11.9 Fields of a uniformly moving charged particle. (*a*) Fields at the observation point P in Fig. 11.8 as a function of time. (*b*) Lines of electric force for a particle at rest and in motion ($\gamma = 3$). The field lines emanate from the *present* position of the charge.

sion of the lines of force in the transverse direction can be viewed as a consequence of the FitzGerald–Lorentz contraction.

11.11 Relativistic Equation of Motion for Spin in Uniform or Slowly Varying External Fields

The effects of a particle's motion on the precession of its spin have already been discussed in Section 11.8 on Thomas precession. Here we exploit the ideas of Lorentz covariance to give an alternative, more elegant discussion leading to what is known as the BMT equation of motion for the spin.* With the magnetic

*Named, not after one of the New York City subway lines, but for V. Bargmann, L. Michel, and V. L. Telegdi, *Phys. Rev. Lett.* **2**, 435 (1959). The equation actually has much earlier origins; Thomas published an equivalent in 1927 (*op. cit.*); Frenkel discussed similar equations contemporaneously; Kramers considered the $g = 2$ equation in the 1930s.